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Coherent states and geodesics: cut locus and conjugate locus

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Abstract

The intimate relationship between coherent states and geodesics is pointed out. For homogenous manifolds in which the exponential from the Lie algebra to the Lie group equals the geodesic exponential, and in particular for symmetric spaces, it is proved that the cut locus of the point 0 is equal to the set of coherent vectors orthogonal to $|0\rangle$. A simple method to calculate the conjugate locus in Hermitian symmetric spaces with significance in the coherent state approach is presented. The results are illustrated on the complex Grassmannian manifold.

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1. Introduction

The coherent states [1] are an excellent interplay of classical and quantum mechanics [2]. The local construction of Perelomov's homogeneous coherent states [3] was globalized, including the Kählerian nonhomogeneous manifolds [4]. Simultaneously, the geometric quantization program [5] furnishes, at least in principle, a tool towards the quantization program of Dirac on differentiable manifolds. Actually, using both the same mathematical objects from complex geometry [6], fibre bundles [7], algebraic geometry [8], ..., the coherent state approach and the geometric quantization are deeply related. In fact, the coherent state approach offers a straightforward recipe for geometric quantization [9].

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However, interesting problems in both these already classical fields have not been yet tackled. One of them is *the relationship between coherent states and geodesics*.

The starting point of this paper is Ref. [10, Remark 3] which expresses in the language of coherent states the property, here called *condition* (A), that for symmetric spaces the geodesics emanating from the point o of the symmetric spaces are given by the exponential exp from the Lie algebra to the Lie group and all the geodesics are obtained in such a way. The aim of this report is to explore further this relationship.

Firstly, let us recall some notions related to geodesics. Let us fix a point p of a complete Riemannian manifold V and a geodesic γ emanating from p. Then the *cut point* [11] of p along γ is the first point on γ such that, for any point r beyond q on γ, there is a shorter geodesic from p to r different from γ. A point q is a *conjugate point* of p along γ if there is a one-parameter family of geodesics from p to q neighbouring γ. Equivalent and precise definitions are given in Section 3. Here we only stress that the importance of the cut loci lies in the fact they inherit topological properties of the manifold V. V may be obtained from $\boldsymbol{CL}_p$ by attaching an n-dimensional cell via the map Exp : $CL_p \rightarrow \boldsymbol{CL}_p$ and $\boldsymbol{CL}_p$ is a strong deformation retract of $V \setminus \{p\}$, where CL (resp. $\boldsymbol{CL}$) denotes the *tangent cut locus* (resp. the *cut locus*).

In this paper it is found that for some homogeneous manifolds there is an intimate connection between the cut locus $\boldsymbol{CL}_0$ of a point on the manifold $\tilde{\boldsymbol{M}}$ corresponding to a fixed coherent vector, say $|0\rangle$, and *the polar divisor* Σ_0, i.e. the locus of coherent vectors orthogonal to $|0\rangle$. The equality

$$\boldsymbol{CL}_0 = \Sigma_0 \tag{1.1}$$

is proved under a technical condition for the manifold, called *condition* (B). It is stressed that condition (B) implies condition (A) and the well-known case of Riemannian symmetric spaces [13–15] is contained as a particular case. Despite the fact that equality (1.1) is proved only for manifolds which satisfy condition (B), this remark is attractive even from pure mathematical point of view, due to the lack of methods to characterize the cut locus as an object of global differential geometry [16]. In this paper we illustrate the results in the case of the complex Grassmannian manifold $G_n(\mathbb{C}^{m+n})$. The cut locus on $G_n(\mathbb{C}^{m+n})$ is well known [13,15].

Another contribution of this article is contained in Theorem 1 which proposes a calculation with significance in the coherent state approach of the conjugate locus for Hermitian symmetric spaces. The connection with the coherent state approach consists in the fact that the parameters Z and B which appear in the geodesic exponential map $Z = Z(B)$ are two different parameters of the coherent states for symmetric spaces. We also illustrate the method in the case of the complex Grassmannian manifold $G_n(\mathbb{C}^{m+n})$. However, the situation in this case is more complicated than in the case of the cut locus. In fact, there are two main contributions in this field. Wong [14] has announced the expression of the conjugate locus in the Grassmannian manifold, while the calculation of the tangent conjugate locus of Sakai [15] shows that Wong's result is incomplete. All these problems are largely discussed elsewhere [17], where another proof of the result of Sakai on the tangent conjugate locus is given and a calculation of the conjugate locus in $G_n(\mathbb{C}^{m+n})$ using Theorem 1 is also given. The only new observation [17] is a geometrical characterization of the part of the conjugate locus not found by Wong [14,15] as consisting of those points of the $G_n(\mathbb{C}^{m+n})$ which have at least two of the stationary angles [18] with a fixed n-plane equal. We have included in Section 4 only the notions necessary to illustrate the results of this paper in the example of $G_n(\mathbb{C}^{m+n})$.

Some of the results included in the present work have been already briefly announced as part of a trial to find a geometrical characterization of Perelomov's construction of coherent state manifold as Kählerian embedding into a projective space [19,20].

The paper is organized as follows. In Section 2 the notation on coherent state manifolds is fixed. The result $\boldsymbol{CL}_0 = \Sigma_0$ and some results on coherent states and conjugate points are proved in Section 3 for manifolds $\tilde{\boldsymbol{M}}$ which satisfy condition (B). Section 4 deals with the Grassmannian manifold.

2. The coherent state manifold and the coherent vector manifold

First we fix the notation referring to the coherent state manifold.

2.1

Let us consider a quantum system with symmetry, i.e. a triplet $(\boldsymbol{K}, G, \pi)$, where π is a unitary irreducible representation of the Lie group G on the Hilbert space $\boldsymbol{K}$. Let us consider the orbit

$$\tilde{\boldsymbol{M}} = \{\tilde{\pi}(g)|\tilde{\psi}_0\rangle \mid g \in G\}, \tag{2.1}$$

where $\tilde{\pi}$ is the projective representation of G induced by π, $|\psi_0\rangle \in \boldsymbol{K}$ is fixed and $\xi : \boldsymbol{K} \to \boldsymbol{PK}$ is the projection $\xi(|\psi\rangle) \equiv |\tilde{\psi}\rangle = \{e^{i\varphi} \mid \psi\rangle \mid \varphi \in \mathbb{R}\}$. Then we have the bijection $\tilde{\xi} : G/K \to \tilde{\boldsymbol{M}}$, $\tilde{\xi}(gK) = \tilde{\pi}(g)|\tilde{\psi}_0\rangle$, where K is the stationary group of the state $|\tilde{\psi}_0\rangle$. The quantum mechanics can be realized as the elementary G-space [21] $(\boldsymbol{PK}, \omega_{\mathrm{FS}}, \rho')$, where ω_{FS} is the Fubini–Study (Kähler) fundamental two-form on the projective space $\boldsymbol{PK}$, and ρ' is the isomorphism of the Lie algebra $\mathfrak{g}$ of G into the algebra of smooth functions on $\boldsymbol{PK}$. The keystone in the coherent state approach is to find a Hilbert space $\boldsymbol{L}$ and a Kählerian embedding $\iota : \tilde{\boldsymbol{M}} \hookrightarrow \boldsymbol{PL}$ [19,22]. Then $\tilde{\boldsymbol{M}}$ is called *coherent state manifold* and $(\tilde{\boldsymbol{M}}, \omega, \rho)$ is a Hamiltonian G-space, with $\omega = \omega_{\mathrm{FS}|\tilde{\boldsymbol{M}}} = \iota^*\omega_{\mathrm{FS}}$, $\rho = \rho'_{|\tilde{\boldsymbol{M}}}$. *Dequantization* means passing on from the dynamical system problem in the initial Hilbert space $\boldsymbol{K}$ to the corresponding one on $\tilde{\boldsymbol{M}}$.

If $|\tilde{\psi}_0\rangle \equiv |j\rangle$, i.e. an (anti-)dominant weight vector for compact connected simply connected Lie groups, then ι is indeed a Kählerian embedding [21] and $\tilde{\boldsymbol{M}}$ coincides with the coadjoint orbit in $\mathfrak{g}^*$ through the root j corresponding to the (anti-)dominant weight vector [23]. So, furnishing both the representation $\pi = \pi_j$ and the Hilbert space $\boldsymbol{K}_j$ of holomorphic sections with base $\tilde{\boldsymbol{M}}$, it is found that $\boldsymbol{L} = \boldsymbol{K}_j^*$, and the Borel–Weil–Bott theorem solves the requantization problem [9,24]. Here E^* denotes the dual of the vectorial space E, i.e. the space of linear functionals on E.

We now briefly discuss the embedding ι for compact complex manifolds $\tilde{M}$. In this case, *the condition for the existence of the embedding ι is equivalent to the requirement for the manifold to be Hodge* [8], *which is the same condition as prequantization in geometric quantization* [25]. For example, in order to have the condition $\omega \in H^2(\tilde{M}, \mathbb{Z})$ fulfilled for Hermitian symmetric spaces it is sufficient that a theorem due to Harish-Chandra [10,5,26] to be satisfied. This theorem in the compact case is just the Borel–Weil–Bott theorem. The Kodaira vanishing theorem replaces the Borel–Weil–Bott theorem, as was already remarked [27] in the context of cohernt states. Let now $\xi_0 : M' \to \tilde{M}$ be a holomorphic line bundle. Another way to express the condition to have the embedding is that the line bundle M' be a positive one, or, equivalently, to be ample (see [28, Theorem. 5.1, p.89]). The last condition means that there exists an integer m_0 such that for $m \geq m_0$, $M \equiv M'^m = \iota^*[1]$. We use the notation $[r] = H^r$, $r \in \mathbb{Z}$, where H is the hyperplane bundle over PL and E^m is the m times tensor product of the bundle E with itself. Here ξ_0 is the positive line bundle appearing in the Kodaira embedding theorem, and the embedding $\iota : \tilde{M} \hookrightarrow PL = \mathbb{CP}^{N-1}$ [29] is

$$\iota \equiv \iota_M : x \to \iota_M(x) = [s_1(x), \ldots, s_N(x)]. \tag{2.2}$$

The line bundle M is furnished by the coherent state approach and is called *coherent vector manifold* [30]. As a consequence of the Kodaira embedding theorem, the Kodaira vanishing [28] theorem implies that in the sum giving the generalized Euler–Poincaré characteristic [8], only the zero term is present, and the dimension of the representation π_j is furnished by the Riemann–Roch–Hirzebruch theorem (cf. [8, Theorem 18.2.2, p.140]). There are situations in which the coherent state approach permits rapid and explicit statements, for example, for flag manifolds, the minimal exponent N appearing in the Kodaira embedding theorem, $\tilde{M} \hookrightarrow CP^{N-1}$, is equal to the Euler–Poincaré characteristic, $N = \chi(\tilde{M})$ [19,30].

The noncompact case is treated similarly by Kobayashi [31], the Hilbert space L being infinite-dimensional. In the construction of Kobayashi, L is the dual of the Hilbert space of square integrable holomorphic n-forms in $\tilde{M}$. If K is the kernel $2n$-form on $\tilde{M} \times \bar{\tilde{M}}$, then the Kähler metric used by Kobayashi is $\mathrm{d}s^2 = \sum(\partial^2 \log K^*/\partial z_i \partial \bar{z}_j)\,\mathrm{d}z_i\,\mathrm{d}\bar{z}_j$, where $K(z,\bar{z}) = K^*(z,\bar{z})\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n$.

The condition (A1) ((A2), respectively, (A3)) in Kobayashi corresponds to the condition of the set of divisors without base points (the differential of ι does not have degenerate points, respectively, the condition (A1) plus the injectivity condition in [29]). We recall that (A1) implies that $\omega_{|\tilde{M}} = \iota^*\omega_{\mathrm{FS}}$, while (A2) and (A3) imply that the application ι is a Kählerian embedding.

We now discuss other cases in which the representation π can be constructed. The condition to have holomorphic discrete series on homogeneous bounded symmetric domains (noncompact Hermitian symmetric spaces) results from the quoted theorem of Harish-Chandra and, more generally, the condition to have discrete series for connected semisimple Lie groups is that rank G = rank K [26]. The problem of structure of homogeneous Kähler manifolds in the context of fundamental conjecture has begun to be handled in connection with the coherent states, especially for the unimodular groups [32].

2.2

In this section we restrict ourselves to coherent state manifolds of flag type, i.e. $\widetilde{\boldsymbol{M}} \approx G/K \approx G^{\mathbb{C}}/P$, where G is a compact connected simply connected semisimple Lie group, $G^{\mathbb{C}}$ is the complexification of G and P is a parabolic subgroup of $G^{\mathbb{C}}$ [10]. The noncompact case is handled similarly, whenever the conditions of the existence of the representation π_j are fulfilled.

Let $W(G) = N(T)/C(T)$ denote the Weyl group associated with G, where $N(T)$ $(C(T))$ is the normalizer (the centralizer) of the Cartan group T. Let $\Sigma \subset N(T)$ be a set of elements such that quotient space $W(G)/W(K)$ is made of the coset classes $\{sC(T)\}W(K)$, $s \in \Sigma$. Then there is an open covering of $\widetilde{\boldsymbol{M}}$ by $(\mathcal{V}_s)_{s\in\Sigma}$, where $\mathcal{V}_s = \pi_j(s)\mathcal{V}_0$, $s \in \Sigma$ [30]. The coherent state vectors corresponding to the points of the neighbourhood $\mathcal{V}_0 \subset \widetilde{\boldsymbol{M}}$ around $Z = 0$ are

$$|Z, j\rangle = \exp \sum_{\varphi\in\Delta_{\mathrm{n}}^{+}} (Z_{\varphi}F_{\varphi}^{+})|j\rangle,\ |\underline{Z}\rangle = \langle Z|Z\rangle^{-1/2}|Z\rangle \in \boldsymbol{M}, \tag{2.3}$$

where $Z \in \mathbb{C}^n$ are local coordinates and n is the dimension of the manifold $\widetilde{\boldsymbol{M}}$. Here

$$F_{\varphi}^{\pm} = \pi^{*\prime}(f_{\varphi}^{\pm}), \quad \varphi \in \Delta_{\mathrm{n}}^{+}, \tag{2.4}$$

$\pi' = \mathrm{d}\pi$, π' is the isomorphism of the Lie algebra $\mathfrak{g}$ of G onto the Lie algebra of operators on $\boldsymbol{K}$, π^* is the group isomorphism $G^{\mathbb{C}} \to \pi^*(G^{\mathbb{C}})$,

$$\pi^*(e^Z) = \exp(\pi^{*\prime}(Z)), \quad Z \in \mathfrak{g}^{\mathbb{C}}, \tag{2.5}$$

$\pi^{*\prime}(\mathfrak{g}^{\mathbb{C}})$ is the complexification of the Lie algebra $\pi'(\mathfrak{g})$, the subindex n (resp. c) abbreviates the noncompact (resp. compact), Δ are the roots and Δ^+ the positive roots.

We also use the notation

$$f_{\varphi}^{\pm} = \begin{cases} k_{\varphi}^{\pm} = \mathrm{i}e_{\pm\varphi} & \text{for } X_{\mathrm{n}}, \\ e_{\varphi}^{\pm} = e_{\pm\varphi} & \text{for } X_{\mathrm{c}}, \end{cases} \tag{2.6}$$

where $e_{\varphi}^{\pm} = e_{\pm\varphi}$ are the part of the Cartan–Weyl base corresponding to $\mathfrak{m}$. Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $\boldsymbol{G}$ and $\mathfrak{k}$ is the Lie algebra of $\boldsymbol{K}$.

The homogeneous symmetric spaces are obtained as

$$X_{\mathrm{n,c}} = \exp \sum_{\varphi\in\Delta_{\mathrm{n}}^{+}} (B_{\varphi}f_{\varphi}^{+} - \bar{B}_{\varphi}f_{\varphi}^{-}) \cdot o, \tag{2.7}$$

where $o = \lambda(e)$, e is the unit element in G and λ is the canonical projection $\lambda : G \to G/K$. Let also the notation

$$|B, j\rangle = \exp \sum_{\varphi\in\Delta_{\mathrm{n}}^{+}} (B_{\varphi}F_{\varphi}^{+} - \bar{B}_{\varphi}F_{\varphi}^{-})|j\rangle, \tag{2.8}$$

$$|B, j\rangle \equiv |\underline{Z, j}\rangle. \tag{2.9}$$

Note that

$$F_{\varphi}^{+}|j\rangle \neq 0, \quad F_{\varphi}^{-}|j\rangle = 0, \quad H_i|j\rangle = j_i|j\rangle, \tag{2.10}$$

where $\varphi \in \Delta_{\mathrm{n}}^{+}$, $H_i = \pi^*(h_i)$, $\{h_i\}$ is a base of the Cartan subalgebra and $i = 1, \ldots, \operatorname{rank} G$.

2.3

We now state more precisely the definition of the coherent vector manifold $\boldsymbol{M}$ corresponding to flag manifolds $\widetilde{\boldsymbol{M}}$. Let $\boldsymbol{M}'$ be the holomorphic line bundle $\boldsymbol{M}' = \xi_0^{-1}(\widetilde{\boldsymbol{M}}) \to G_{\mathrm{c}}/P$ associated by the holomorphic character $\chi = \chi_j$ of P to the principal bundle $P \to G^{\mathbb{C}} \to G^{\mathbb{C}}/P$, i.e. the line bundle obtained identifying $(g, \chi(p)w)$ with (gp, w), where $p \in P$, $w \in \mathbb{C}$.

In fact, if $(\boldsymbol{M}', \omega, J)$ is the compact Kähler manifold (J is the complex structure, $J = ad(Z)|_{\mathfrak{m}}$ and Z is the central element of the Lie algebra $\mathfrak{k}$), then $(\boldsymbol{M}', \nabla, \iota^*(h))$ is a quantization bundle over $\widetilde{\boldsymbol{M}}$ [5], where h is the Hermitian form on the hyperplane line bundle [1] over $\boldsymbol{PL}$. Then, on the tautological line bundle $[-1]$, h is given by $h : z \to |z|^2$. Also $curv(\nabla) = -2\pi\mathrm{i}\omega$, so $\omega \in c_1(\boldsymbol{M}') = [\omega]_{\text{de Rham}}$.

If $\varphi_i : \mathcal{V}_i \times \mathbb{C} \to \xi_0^{-1}(\mathcal{V}_i)$ is the local trivialization of the holomorphic line bundle $\boldsymbol{M}' \to \widetilde{\boldsymbol{M}}$, then a global section is given by

$$|s_i(m)\rangle = (g_i(Z_i), f_{s_i}(Z_i)) = (g_i(Z_i), \langle s_i|Z_i\rangle), \tag{2.11}$$

where $m = g_i(Z_i) \in \mathcal{V}_i$ are matrix elements determined by the local coordinates Z_i. Then the scalar product on the line bundle $\boldsymbol{M}' \to \widetilde{\boldsymbol{M}}$ is given by [9,30]

$$\begin{aligned} \langle s_i|s_i'\rangle &= \int_{\widetilde{\boldsymbol{M}}} h_X(s_i(X), s_i'(X)) \frac{\omega^n(X)}{n!} \\ &= \langle f_{s_i}, f_{s_i'}\rangle = \int_{\widetilde{\boldsymbol{M}}} h_X(f_{s_i}(X), f_{s_i'}(X)) \frac{\mathrm{d}\mu(X)}{\langle X|X\rangle}, \end{aligned} \tag{2.12}$$

where $\mathrm{d}\mu(X)$ is the Haar measure on $\widetilde{\boldsymbol{M}} \approx G^{\mathbb{C}}/P$.

The scalar product in (2.12) is also a Hermitian scalar product of sections with base $\widetilde{\boldsymbol{M}}$ in the $D_{\widetilde{\boldsymbol{M}}}$- module of differentiable operators on $\widetilde{\boldsymbol{M}}$ [10].

When both the dequantization and the requantization can be done, the Hilbert space $\boldsymbol{K}_j$ attached to the representation π_j and the initial $\boldsymbol{K}$ are isomorphic [9,27].

3. The cut locus and coherent states

In this section we shall be concerned with various aspects of the relationship between geodesics and coherent states. We briefly review some definitions used in Section 1.

3.1

Let V be compact Riemannian manifold of dimension n, $p \in V$ and let Exp_p be the (geodesic) exponential map at the point p. Let C_p denote the set of vectors $X \in V_p$ (the tangent space at $p \in V$) for which $\mathrm{Exp}_p X$ is singular. A point q in V (V_p) *is conjugate to* p if it is in $\boldsymbol{C}_p = \mathrm{Exp}\, C_p$ (C_p) [12] and $\boldsymbol{C}_p$ (resp. C_p) is called *the conjugate locus* (resp. *tangent conjugate locus*) of the point p.

Let $q \in V$. The point q is in the *cut locus* $\boldsymbol{CL}_p$ of $p \in V$ if it is the nearest point to $p \in V$ on the geodesic joining p with q, beyond which the geodesic ceases to minimize its arc length [11]. More precisely, let $\gamma_X(t) = \mathrm{Exp}\, tX$ be a geodesic emanating from $\gamma_X(0) = p \in V$, where X is a unit vector from the unit sphere S_p in V_p. t_0X (resp. $\mathrm{Exp}\, t_0X$) is called a *tangential cut point* (*resp. cut point*) of p along $t \to \mathrm{Exp}\, tX$ $(0 \leq t \leq s)$ if the geodesic segment joining $\gamma_X(0)$ and $\gamma_X(t)$ is a minimal geodesic for any $s \leq t_0$ but not for any $s > t_0$.

Let us define the function $\mu : S_p \to \mathbb{R}^+ \cup \infty$, $\mu(X) = r$, if $q = \mathrm{Exp}\, rX \in \boldsymbol{CL}_p$, and $\mu(X) = \infty$ if there is no cut point of p along $\gamma_X(t)$. Setting $I_p = \{tX,\ 0 < t < \mu(X)\}$, then $\boldsymbol{I}_p = \mathrm{Exp}\, I_p$ is called the *interior set at* p. Then:

(1) $\boldsymbol{I}_p \cap \boldsymbol{CL}_p = \emptyset$, $V = \boldsymbol{I}_p \cup \boldsymbol{CL}_p$, the closure $\bar{\boldsymbol{I}}_p = V$, and $\dim \boldsymbol{CL}_p \leq n - 1$;

(2) I_p is a maximal domain containing $0 = 0_p \in V_p$ in which Exp_p is a diffeomorphism and $\boldsymbol{I}_p$ is the largest open subset of V in which a normal coordinate system around p can be defined.

The relative position of $\boldsymbol{CL}_0$ and $\boldsymbol{C}_0$ given in [11, Theorem 7.1, p.97] is reproduced below.

Let the notation $\gamma_t = \gamma_X(t)$. Let γ_r be the cut point of γ_0 along a geodesic $\gamma = \gamma_t$, $0 \leq t < \infty$. Then, at least one (possibly both) of the following statements holds:

(1) γ_r is the first conjugate point of γ_0 along γ;

(2) there exists, at least, two minimizing geodesics from γ_0 to γ_r.

Crittenden [33] has shown that for the case of simply connected symmetric spaces, the cut locus is identified to the first conjugate point. Generally, the situation is more complicated [34,35].

Here are simple examples of cut loci. For the sphere S^n, the cut locus of a point reduces to the antipodal point, while the tangent cut locus CL is the sphere of radius π with centre at the origin of the tangent space. For $\mathbb{CP}^n$, CL is also the sphere of radius π with centre at the origin of the tangent space to $\mathbb{CP}^n$ at the given point, while $\boldsymbol{CL}$ is the hyperplane at infinity $\mathbb{CP}^{n-1}$. Except few situations, e.g. the ellipsoid, even for low-dimensional manifolds as the (asymmetric Berger's spheres) S^3, $\boldsymbol{CL}$ is not known explicitly. Helgason [12] has shown that the cut locus of a compact connected Lie group endowed with a bi-invariant Riemannian metric is stratified, i.e. it is the disjoint union of smooth submanifolds of V. This situation will be illustrated in the case of complex Grassmannian manifold. Using a geometrical method, Wong [13,14,36] has studied conjugate loci and cut loci of the Grassmannian manifolds emphasizing also their stratification. Sakai [37] has found out the cut locus of the connected compact symmetric manifold $V = U(n)/O(n)$, which has $\pi_1(V) \cong \mathbb{Z}$. By refining the results of Ch. VII, Section 5 "control over singular set" from [12], Sakai [15,38] studied the cut locus of a point in a compact symmetric space which is not necessarily simply connected and showed that it is determined by the cut locus of a maximal totally geodesic flat torus of V. Takeuchi [39] has also proved the stratified structure of $\boldsymbol{CL}$ and $\boldsymbol{C}$ for compact symmetric manifolds. For other references see [16]. However, the expression of the conjugate locus as subset of the Grassmannian manifold is not known explicitly. This problem is largely discussed elsewhere [17]. In Section 4 of this paper we shall only collect the main results of this problem.

Most considerations in this section concern only manifolds with the property

(A) $\quad \mathrm{Exp}|_0 = \lambda \circ \exp|_{\mathfrak{m}}$.

Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the orthogonal decomposition with respect to the B-form as explained in (B) below, $\mathrm{Exp}_p : \widetilde{\boldsymbol{M}}_p \to \widetilde{\boldsymbol{M}}$ is the geodesic exponential map (cf. [12, p.33]) and $\exp : \mathfrak{g} \to G$.

In fact, (A) expresses that *the geodesics in $\widetilde{\boldsymbol{M}}$ are images of one-parameter subgroups of $\widetilde{\boldsymbol{M}} \approx G/K$*. The symmetric spaces have property (A) (cf.[12, Theorem 3.3, p.208]).

We shall also be concerned with manifolds $\widetilde{\boldsymbol{M}}$ satisfying the following condition:

(B) On the Lie algebra $\mathfrak{g}$ of G there exists an $Ad(G)$-invariant, symmetric, nondegenerate bilinear form B such that the restriction of B to the Lie algebra $\mathfrak{k}$ of K is likewise nondegenerate.

We point out that *if the homogeneous space $\widetilde{\boldsymbol{M}} \approx G/K$ satisfies* (B), *then it also satisfies* (A) (cf. [11, Ch. X, Corollaries 2.5 and 3.6, Theorem 3.5]). Indeed, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the orthogonal decomposition relative to the B-form on $\mathfrak{g}$, then $\mathfrak{m}$ is canonically identified with the tangent space at o, $\widetilde{\boldsymbol{M}}_o$. (B) implies a (possibly indefinite) G-invariant metric on $\widetilde{\boldsymbol{M}}$. It follows that G/K is reductive, i.e. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. If (B) is true, then $\widetilde{\boldsymbol{M}}$ is naturally reductive (see [11, p.202]) and (A) is also satisfied. The symmetric spaces satisfy besides the conditions of reductive spaces, the condition $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and, of course, (A) is satisfied too (see [11, Ch. XI, Theorem 3.2]).

Thimm [40] furnishes as another examples of homogeneous spaces satisfying (B), besides the symmetric spaces, the Lie groups with bi-invariant metric and the normal homogeneous spaces (i.e. B is positive definite). Kowalski [41] studied generalized symmetric spaces still satisfying condition (A). See also [42].

3.2

We now recall that in Ref. [10] we did the following remark, which is in fact Cartan's theorem (see e.g. [12, Theorem 3.3, p.208]) on geodesics on symmetric spaces expressed in the coherent state setting.

Remark 1. *The vector $|tB, j\rangle = \exp \pi_j^{*\prime}(tB)|j\rangle \in \boldsymbol{M}$, $B \in \mathfrak{m}$, describes trajectories in $\boldsymbol{M}$ corresponding to the image in the manifold of coherent states $\widetilde{\boldsymbol{M}} \hookrightarrow \boldsymbol{PL}$ of geodesics through the identity coset element on the symmetric space $X \approx G/K$. The dependence $Z(t) = Z(tB)$ appearing when one passes from Eqs. (2.8) to (2.3) describes in $\mathcal{V}_0$ a geodesic.*

We shall reformulate Remark 1 in a way very useful even for practical calculations. The proof presented below, true in the particular case of Hermitian symmetric spaces, implies also Theorem 1.

Remark 2. *For an n-dimensional manifold $X \approx G/K$ which has Hermitian symmetric space structure, the parameters B_φ in formula (2.8) of normalized coherent states are normal coordinates in the normal neighbourhood $\mathcal{V}_0 \approx \mathbb{C}^n$ around the point $Z_\varphi = 0$ on the manifold X.*

Proof. The Harish-Chandra embedding theorem can be used (cf. e.g. [26]; see also [10] for the present context). This theorem asserts that the map $M^+ \times K^{\mathbb{C}} \times M^- \to G^{\mathbb{C}}$ given by $(m^+, k, m^-) \to m^+ k m^-$ is a complex analytic diffeomorphism onto an open dense subset of $G^{\mathbb{C}}$ that contains G_n. Let $\mathfrak{m}^\pm$ be the $\pm i$ eigenspaces of J and $M^\pm$ the (unipotent, Abelian) subgroups of $G^{\mathbb{C}}$ corresponding to $\mathfrak{m}^\pm$. Then, in particular, $b : \mathfrak{m}^+ \to X_c = G^{\mathbb{C}}/P$, $b(X) = \exp(X)P$, is a complex analytic diffeomorphism of $\mathfrak{m}^+$ onto a dense subset of X_c (that contains X_n) and the remark follows because the requirement (A) is fulfilled for the symmetric spaces. □

Another way to reformulate Remark 1 is the following.

Theorem 1. *Let $\widetilde{\boldsymbol{M}}$ be a coherent state manifold with Hermitian symmetric space structure, parametrized in $\mathcal{V}_0$ around $Z = 0$ as in Eqs. (2.3), (2.8). Then the conjugate locus of the point o is obtained vanishing the Jacobian of the exponential map $Z = Z(B)$ and the corresponding transformations of the chart from $\mathcal{V}_0$.*

Proof. The proof is contained in Remark 2. *The dependence $Z = Z(B)$*, with $B \in \mathfrak{m}^+$, and Z parametrizing $\widetilde{\boldsymbol{M}}$, obtained passing from Eqs. (2.8) to (2.3) using the relations (2.10) (the Baker–Campbell–Hausdorff formulas) [10], *expresses* in fact *the geodesic exponential* $\mathrm{Exp}_0 : \widetilde{\boldsymbol{M}}_0 \to \widetilde{\boldsymbol{M}}$. □

The situation is very transparent in the case of the complex Grassmannian manifold $X_c = G_n(\mathbb{C}^{n+m}) = SU(n+m)/S(U(n) \times U(m))$ and his noncompact dual $X_n = SU(n,m)/S(U(n) \times U(m))$. There [10]

$$
\begin{aligned}
X_{n,c} &= \exp\begin{pmatrix} 0 & B \\ \pm B^* & 0 \end{pmatrix} o = \begin{pmatrix} \mathrm{co}\sqrt{BB^*} & B\dfrac{\mathrm{si}\sqrt{B^*B}}{\sqrt{B^*B}} \\ \pm\dfrac{\mathrm{si}\sqrt{B^*B}}{\sqrt{B^*B}}B^* & \mathrm{co}\sqrt{B^*B} \end{pmatrix} o \\
&= \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1 \mp ZZ^*)^{1/2} & 0 \\ 0 & (1 \mp Z^*Z)^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm Z^* & 1 \end{pmatrix} o \\
&= \exp\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} P,
\end{aligned} \tag{3.1}
$$

where B^* denotes the Hermitian conjugate of the matrix B. co is an abbreviation for the circular cosine cos (resp. the hyperbolic cosine coh) for $X_{\rm c}$ (resp. $X_{\rm n}$) and similarly for si. The $-$ (resp. $+$) sign in the above equation corresponds to the compact (resp. noncompact) X.

Here Z and B are the $n \times m$ matrices related by the relation

$$Z = B\frac{\mathrm{ta}\sqrt{B^*B}}{\sqrt{B^*B}}, \tag{3.2}$$

and ta is an abbreviation for the hyperbolic tangent tgh (resp. the circular tangent tg) for $X_{\rm n}$ (resp. $X_{\rm c}$). The dependence $Z = Z(B)$ describes in fact $\mathrm{Exp} : G_n(\mathbb{C}^{n+m})_e \to G_n(\mathbb{C}^{n+m})$ in $\mathcal{V}_0$. Indeed, the equation of geodesics for $X_{\rm c,n}$ is [17]:

$$\frac{\mathrm{d}^2Z}{\mathrm{d}t^2} - 2\epsilon\frac{\mathrm{d}Z}{\mathrm{d}t}Z^+(1+\epsilon ZZ^+)^{-1}\frac{\mathrm{d}Z}{\mathrm{d}t} = 0\,, \tag{3.3}$$

where $\epsilon = 1$ (resp. -1) for $X_{\rm c}$ (resp. $X_{\rm n}$). It is easy to see that (3.2) satisfies (3.3) with the initial condition $\dot{Z}(0) = B$.

Z and B in Eq. (3.2) of geodesics are in the same time the parameters describing the coherent states in the parametrization given by Eq. (2.3) and (2.8), respectively.

3.3

Firstly, let us introduce a notation for the *polar divisor* of $|0\rangle \in \boldsymbol{M}$:

$$\Sigma_0 = \{|\psi\rangle \,|\, |\psi\rangle \in \boldsymbol{M},\ \langle 0|\psi\rangle = 0\}\,. \tag{3.4}$$

This denomination is inspired from that one used by Wu [43] in the case of the Grassmannian manifold.

We shall prove the following theorem.

Theorem 2. *Let $\tilde{\boldsymbol{M}}$ be a homogeneous manifold $\tilde{\boldsymbol{M}} \approx G/K$. Suppose that there exists a unitary irreducible representation π_j of G such that in a neighbourhood $\mathcal{V}_0$ around $Z = 0$ the coherent states are parametrized as in Eq.* (2.3). *Then the manifold $\tilde{\boldsymbol{M}}$ can be represented as the disjoint union*

$$\tilde{\boldsymbol{M}} = \mathcal{V}_0 \cup \Sigma_0. \tag{3.5}$$

Moreover, if condition (B) *is true, then*

$$\Sigma_0 = \boldsymbol{CL}_0 \tag{3.6}$$

and

$$\mathcal{V}_0 = \boldsymbol{I}_0. \tag{3.7}$$

Proof. We can take $|\psi\rangle = |\psi(Z)\rangle \in \boldsymbol{M}$ such that the parameters Z are in $\mathbb{C}^n$ as in formula (2.3). Now, the second relation (2.10) implies that $\langle 0|\psi\rangle = 1$ for $|\psi\rangle \in \xi_0^{-1}(\mathcal{V}_0)$. It follows that the equation

$$\cos\theta = 0, \tag{3.8}$$

where

$$\cos\theta = \frac{|\langle 0|\psi\rangle|}{\|0\|^{1/2}\|\psi\|^{1/2}} = \|\psi\|^{-1/2}, \tag{3.9}$$

does not have solutions for $|\psi\rangle \in \xi_0^{-1}(\mathcal{V}_0)$, and the representation (3.5) follows.

To prove relation (3.6) if (B) is true, use is made of Theorem 7.4 and the subsequent remark at p.100 from Ref. [11] reproduced in the beginning of this section in an enriched version. The theorem essentially says that any Riemannian manifold $\tilde{\boldsymbol{M}}$ is the disjoint union of the cut locus (closed cell) and the largest open cell of $\tilde{\boldsymbol{M}}$ in which normal coordinates can be defined. But $Z \in \mathbb{C}^n$ for points of $\mathcal{V}_0$ corresponding to the largest normal coordinates $B \in \mathfrak{m}$, because (B) implies (A). □

Further we shall prove a corollary of Theorem 2. This is related to the angle θ appearing in Eq. (3.9).

Firstly, let us introduce the (Hermitian elliptic) Cayley distance [44] in the projective space. Let $(\cdot,\cdot)$ be the scalar product in $\boldsymbol{K}$. If $\xi : \boldsymbol{K}\setminus\{0\} \to \boldsymbol{PK}$ is the natural projection $\xi : \omega \to [\omega]$, then the Cayley distance is

$$d_c([\omega'],[\omega]) = \arccos\frac{|(\omega',\omega)|}{\|\omega'\|\|\omega\|}\ . \tag{3.10}$$

The infinite-dimensional case is argued in Ref. [31]. Before proving the corollary, we shall present the following remark.

Remark 3 (Geometrical significance of transition amplitudes for coherent states).
Let $|\underline{Z}\rangle \in \boldsymbol{M}$, $Z \in \mathcal{V}_0$ as in (2.3) and $\iota : \tilde{\boldsymbol{M}} \hookrightarrow \boldsymbol{PL}$ the embedding of the homogeneous coherent state manifold into the projective space. Then the angle $\theta = \theta(Z, Z')$ defined by

$$\theta \equiv \arccos|\langle\underline{Z'}|\underline{Z}\rangle| \tag{3.11}$$

is equal to geodesic distance joining $\iota(Z)$ and $\iota(Z')$,

$$\theta = d_c(\iota(Z'),\iota(Z)). \tag{3.12}$$

More generally, the (Cauchy) formula is true:

$$\langle\underline{Z'}|\underline{Z}\rangle = \frac{(\iota(Z'),\iota(Z))}{\|\iota(Z')\|\ \|\iota(Z)\|}. \tag{3.13}$$

Proof. Relation (3.13) is an immediate consequence of the fact that the homogeneous complex analytic line bundle $\boldsymbol{M}$ over $\tilde{\boldsymbol{M}}$ is *projectively induced* (see [8, p.139]), i.e. the coherent state manifold $\boldsymbol{M}$ is the pull-back of the hyperplane bundle $H = [1]$ on $\boldsymbol{PL}$, i.e. $\boldsymbol{M} = \iota^*[1]$ [27]. □

The denomination of Eq. (3.13) as the Cauchy formula is due to the fact that for the Plücker embedding of the Grassmannian manifold this formula is nothing else than the (Binet–) Cauchy formula [45].

Corollary 1. *Suppose that $\widetilde{M}$ is a homogeneous manifold satisfying* (B) *and admitting the embedding $\iota : \widetilde{M} \hookrightarrow PL$. Let $0, Z \in \widetilde{M}$. Then $Z \in CL_0$ iff the Cayley distance between the images $\iota(0), \iota(Z) \in PL$ is $\frac{1}{2}\pi$,*

$$d_c(\iota(0), \iota(Z)) = \tfrac{1}{2}\pi. \tag{3.14}$$

Proof. The corollary follows combining Remark 3 and Theorem 2. □

4. An example of the complex Grassmannian manifold

The results of Section 3 will be illustrated in the example of the complex Grassmannian manifold. The calculation of the cut locus on $G_n(\mathbb{C}^{n+m})$ was announced by Wong [13] and now more proofs (see e.g. [15,31]) are available. Also Wong [14] has announced the conjugate locus on the Grassmannian manifold, but, as far as I know, the proof has not been published. Even more, the results of Wong on conjugate locus on Grassmannian manifold were contested by Sakai [15], who showed that the result of Wong is incomplete.

The explicit calculation of the conjugate locus in the manifold using Theorem 1 is presented elsewhere [17]. Another proof of the results of Sakai referring to the tangent conjugate locus is also presented there. Here we just indicate the parameters appearing in the calculation in order to illustrate how the assertions of Section 3 referring to the cut locus and conjugate locus work in a concrete example. However, we do not have an explicit expression for the part of the conjugate locus lost by Wong and only a geometrical characterization in terms of the stationary angles.

4.1

Firstly we fix the notation concerning the geometric construction of coherent state manifold when $\widetilde{M}$ is the complex Grassmannian manifold (the manifold of Slater determinants [46]).

Let $\boldsymbol{O}$ be the n-plane passing through the origin of $\mathbb{C}^N (N = n + m)$ corresponding to $Z = 0$ in $\mathcal{V}_0 \subset G_n(\mathbb{C}^N)$ in the representation (2.3). Then $Z \in \mathcal{V}_0 \approx \mathbb{C}^{n\times m}$ iff there are n vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n \in \mathbb{C}^N$ such that

$$Z = \boldsymbol{z}_1 \wedge \cdots \wedge \boldsymbol{z}_n \neq 0\,. \tag{4.1}$$

We use the Pontrjagin coordinates. Fixing the canonical basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ for $\mathbb{C}^N$, then

$$\boldsymbol{z}_i = \boldsymbol{e}_i + \sum_{\alpha=n+1}^{N} Z_{i\alpha}\boldsymbol{e}_\alpha, \quad i = 1, \ldots, n. \tag{4.2}$$

If the weight j is taken as [10]

$$j = (\underbrace{1, \ldots, 1}_{n}, \underbrace{0, \ldots, 0}_{m}), \tag{4.3}$$

then we have the *equality* [17] *of the scalar product* $\langle\cdot|\cdot\rangle$ *of coherent vectors from* $\boldsymbol{M}$ *and of the Hermitian scalar product* $((\cdot,\cdot))$ *in the holomorphic line bundle* det* [6]:

$$\langle Z'|Z\rangle = ((\hat{Z}',\hat{Z})) = \det((z'_i, z_j))_{i,j=1,\dots,n} = \det(1_n + ZZ'^*)\ . \tag{4.4}$$

We have used the notation

$$\hat{Z} = (1_n, Z), \tag{4.5}$$

where Z is an $n\times m$ matrix and 1_n is the unity $n\times n$ matrix.

So, *the parameters* Z *in formula* (2.3) *for the Grassmannian manifold of coherent states are the Pontrjagin* [47] *coordinates* Z *in formula* (4.2).

Let us also introduce the Plücker coordinates $Z^{i_1\dots i_n}$, i.e.

$$Z = \sum_{1\le i_1<\dots<i_n\le N} Z^{i_1\cdots i_n} e_{i_1}\wedge\dots\wedge e_{i_n}. \tag{4.6}$$

Let $\iota : G_n(\boldsymbol{K}) \hookrightarrow \boldsymbol{PL}$ be the Plücker embedding, where $\boldsymbol{K} = \mathbb{C}^N$, $\boldsymbol{L} = \mathbb{C}^{*N(m)}$, $N(m) = \binom{N}{n} - 1$. Using the notation of Section 2 for $X = X_c = G_n(\mathbb{C}^N)$, then $\boldsymbol{M}' = \boldsymbol{M}$, that is $m_0 = 1$ in $\boldsymbol{M}'^{m_0} = \boldsymbol{M}$ (i.e. the line bundle det* is not only ample, but very ample [28]) and $\boldsymbol{M} = \iota^*[1]$, where [1] is the hyperplane section H in $\boldsymbol{L}$.

The (Binet–) Cauchy formula [45] invoked in Eq. (3.13) reads explicitly

$$\det((z'_i, z_j))_{i,j=1,\dots,n} = \sum_{1\le i_1<\dots<i_n\le N} Z^{i_1\dots i_n}\overline{Z}'^{i_1\dots i_n}\ . \tag{4.7}$$

4.2

We now fix the notation referring to the Schubert varieties.

Let the sequences of integers

$$\omega = \{0 \le \omega(1) \le \dots \le \omega(n) \le m\}, \tag{4.8}$$

$$\sigma(i) = \omega(i) + i, \quad i = 1,\dots,n. \tag{4.9}$$

The Schubert varieties are defined as [47]

$$Z(\omega) = \left\{X \in G_n(\mathbb{C}^{n+m}) \mid \dim(X\cap\mathbb{C}^{\sigma(i)}) \ge i\right\}. \tag{4.10}$$

$Z(\omega)$ are closed cells in the Grassmannian manifold. The "jumps" sequence [48] is introduced as

$$I_\omega = \{i_0 < i_1 < \dots < i_{l-1} < i_l = n\}, \tag{4.11}$$

where

$$\omega(i_h) < \omega(i_{h+1}),\quad \omega(i) = \omega(i_{h-1}),\quad i_{h-1} < i \le i_h,\quad h = 1,\dots,l. \tag{4.12}$$

Let us consider the subset of generic elements of $Z(\omega)$ [47]:

$$Z'(\omega) = \left\{X \subset G_n(\mathbb{C}^{n+m}) | \dim(X\cap\mathbb{C}^{\sigma(i_h)}) = i_h,\ i_h\in I_\omega\right\}. \tag{4.13}$$

The condition to get generic elements Z of $Z(\omega)$, $Z \in \mathcal{V}_0 \cap Z(\omega) \subset Z'(\omega)$, is [47,49]:

$$Z_{ij} = 0, \quad j > \omega(i), \quad i = 1, \ldots, n. \tag{4.14}$$

Let also the notation

$$V_l^p = \left\{ Z \subset G_n(\mathbb{C}^{n+m}) | \dim(Z \cap \mathbb{C}^p) \geq l \right\}, \tag{4.15}$$

$$W_l^p = V_l^p - V_{l+1}^p = \left\{ Z \subset G_n(\mathbb{C}^{n+m}) \mid \dim(Z \cap \mathbb{C}^p) = l \right\}, \tag{4.16}$$

$$\omega_l^p = (\underbrace{p-l, \ldots, p-l}_{l}, \underbrace{m, \ldots, m}_{n-l}). \tag{4.17}$$

Then [14,36,17]

$$V_l^p = Z(\omega_l^p); \quad W_l^p = Z'(\omega_l^p). \tag{4.18}$$

4.3

We now briefly recall some notions referring to the stationary angles.

Let Z', Z be two n-planes of $G_n(\mathbb{C}^{n+m})$ given as in Eq. (4.1). Then the (n) *stationary angles* (see [18] for the real case), *of which most* $r = \min(m, n)$ *are nonzero*, are defined as the *stationary* angles $\theta \in [0, \frac{1}{2}\pi]$ between the vectors

$$\boldsymbol{a} = \sum_{i=1}^{n} a_i \boldsymbol{z}'_i, \qquad \boldsymbol{b} = \sum_{i=1}^{n} b_i \boldsymbol{z}_i, \tag{4.19}$$

where

$$\cos\theta = \frac{|(\boldsymbol{a}, \boldsymbol{b})|}{|\boldsymbol{a}||\boldsymbol{b}|}. \tag{4.20}$$

We recollect the following two lemmas [17,18,50,51].

Lemma 1. *The squares* $\cos^2\theta_i$ *of the stationary angles between the n-planes* Z, Z' *with* $((Z, Z')) \neq 0$ *are given as the eigenvalues of a matrix* W *which, for* $Z, Z' \in \mathcal{V}_0$, *is*

$$W = (1 + ZZ^+)^{-1}(1 + ZZ'^+)(1 + Z'Z'^+)^{-1}(1 + Z'Z^+). \tag{4.21}$$

Lemma 2. *Let* θ *be the angle defined by the Hermitian scalar product in the following equation:*

$$\cos\theta(Z', Z) \equiv \frac{|((Z', Z))|}{\|Z'\|\|Z\|} = \frac{|\det(1 + ZZ'^+)|}{|\det(1 + ZZ^+)|^{1/2}|\det(1 + Z'Z'^+)|^{1/2}}, \tag{4.22}$$

d_c *the Cayley distance and* $\theta_1, \ldots, \theta_n$ *the stationary angles. Then*

$$\cos\theta(Z, Z') = \cos d_c(\iota(Z'), \iota(Z)) = \cos\theta_1 \cdots \cos\theta_n. \tag{4.23}$$

It can be proved [17] that the eigenvalues of W appear also in the expression of the distance on the complex Grassmannian manifold (see also [52]).

Note also that if expression (3.2) of the dependence $Z = Z(B)$ is introduced in the formula of the distance between the points $Z = 0$ and $Z \in V_0$ on the Grassmannian manifold, then

$$d^2 = \sum |B_{ij}|^2. \tag{4.24}$$

The last equation expresses the fact that the parameters B in Eq. (2.8) of coherent states are indeed the normal coordinates as it is asserted in Remark 2.

4.4

We present below the cut locus and the conjugate locus for $G_n(\mathbb{C}^{m+n})$. $\boldsymbol{O}^{\perp}$ denotes the orthogonal complement of the n-plane $\boldsymbol{O}$ in $\mathbb{C}^N$.

Remark 4 (Wong [13]). *The cut locus of the point O is given by*

$$\begin{aligned} CL_0 = \Sigma_0 &= V_1^m = Z(\omega_1^m) = Z(m-1, m, \ldots, m) \\ &= \left\{ X \subset G_n(\mathbb{C}^{n+m}) |\, \dim(X \cap \boldsymbol{O}^{\perp}) \geq 1 \right\}. \end{aligned} \tag{4.25}$$

The cut locus in $G_n(\mathbb{C}^{m+n})$ is given by those n-planes which have at least one of the stationary angles $\frac{1}{2}\pi$ with the n-plane $\boldsymbol{O}$.

Proof. An immediate proof can be obtained using the results of Wu referring to the polar divisor Σ_0 on the Grassmannian manifold (see [43, Ch. 1]) and the theorems characterizing the canonical (universal, det) bundle on $G_n(\mathbb{C}^N)$ (see especially [7, Ch. 7, Proposition 3.3]), which are particularizations of the representation in Theorem 2. □

The following theorem summarizes the known facts about the tangent conjugate locus and conjugate locus in $G_n(\mathbb{C}^{m+n})$ [13,15,17]. The relevant fact for the present paper is that the conjugate locus can be calculated using Theorem 1.

Theorem 3. *The tangent conjugate locus C_0 of the point $\boldsymbol{O} \in G_n(\mathbb{C}^{m+n})$ is given by*

$$C_0 = \bigcup_{k,p,q,i} ad\, k(t_i H), \quad i = 1, 2, 3; \quad 1 \leq p < q \leq r, \quad k \in K, \tag{4.26}$$

where the vector $H \in \mathfrak{a}$ is normalized,

$$H = \sum_{i=1}^{r} h_i D_{i\, n+i}, \quad h_i \in \mathbb{R}, \quad \sum h_i^2 = 1\,. \tag{4.27}$$

The parameters t_i, $i = 1, 2, 3$, in Eq. (4.26) are:

$$t_1 = \frac{\lambda\pi}{|h_p \pm h_q|}, \quad \textit{multiplicity } 2;$$

$$t_2 = \frac{\lambda\pi}{2|h_p|}, \quad \textit{multiplicity } 1; \tag{4.28}$$

$$t_3 = \frac{\lambda\pi}{|h_p|}, \quad \textit{multiplicity } 2|m-n|; \quad \lambda \in \mathbb{Z}^\star.$$

The conjugate locus of $\boldsymbol{O}$ in $G_n(\mathbb{C}^{m+n})$ is given by the union

$$C_0 = \boldsymbol{C}_0^W \cup \boldsymbol{C}_0^I. \tag{4.29}$$

The following relations are true:

$$\boldsymbol{C}_0^I = \exp \bigcup_{k,p,q} Ad\, k(t_1 H), \tag{4.30}$$

$$\boldsymbol{C}_0^W = \exp \bigcup_{k,p} Ad\, k(t_2 H), \tag{4.31}$$

i.e. exponentiating the vectors of the type $t_1 H$ we get the points of $\boldsymbol{C}_0^I$ for which at least two of the stationary angles with $\boldsymbol{O}$ are equal, while the vectors of the type $t_2 H$ are sent to the points of $\boldsymbol{C}_0^W$ for which at least one of the stationary angles with $\boldsymbol{O}$ is 0 or $\frac{1}{2}\pi$.

The $\boldsymbol{C}_0^W$ part of the conjugate locus is given by the disjoint union

$$\boldsymbol{C}_0^W = \begin{cases} V_1^m \cup V_1^n, & n \le m, \\ V_1^m \cup V_{n-m+1}^n, & n > m, \end{cases} \tag{4.32}$$

where

$$V_1^m = \begin{cases} \mathbb{C}P^{m-1} & \textit{for } n = 1, \\ W_1^m \cup W_2^m \cup \cdots \cup W_{r-1}^m \cup W_r^m, & 1 < n, \end{cases} \tag{4.33}$$

$$W_r^m = \begin{cases} G_r(\mathbb{C}^{\max(m,n)}), & n \neq m, \\ \boldsymbol{O}^\perp, & n = m, \end{cases} \tag{4.34}$$

$$V_1^n = \begin{cases} W_1^n \cup \cdots \cup W_{r-1}^n \cup \boldsymbol{O}, & 1 < n \le m, \\ \boldsymbol{O}, & n = 1, \end{cases} \tag{4.35}$$

$$V_{n-m+1}^n = W_{n-m+1}^n \cup W_{n-m+2}^n \cup \cdots \cup W_{n-1}^n \cup \boldsymbol{O}, \quad n > m. \tag{4.36}$$

Proof (sketch). The tangent conjugate locus C_0 for $G_n(\mathbb{C}^{m+n})$ in the case $n \le m$ was obtained by Sakai [15]. Sakai has observed that Wong's result on the conjugate locus in the manifold is incomplete, i.e. $\boldsymbol{C}_0^W \subset \boldsymbol{C}_0$ but $\boldsymbol{C}_0^W \overset{\subset}{\neq} \boldsymbol{C}_0 = \exp C_0$. The proof of Sakai consists in solving the eigenvalue equation $R(X, Y^i)X = e_i Y^i$ which appears when solving the Jacobi equation, where the curvature for the symmetric space $X_c = G_c/K$ at o is simply $R(X,Y)Z = [[X,Y],Z]$, $X, Y, Z \in \mathfrak{m}_c$. Then $q = \mathrm{Exp}_0\, tX$ is conjugate to o if $t = \pi\lambda/\sqrt{e_i}$, $\lambda \in \mathbb{Z}^\star \equiv \mathbb{Z}\backslash\{0\}$.

Above $\mathfrak{a}$ is the Cartan subalgebra of the symmetric pair $(SU(n+m), S(U(n) \times U(m)))$ [12,15,17] consisting of vectors of the form (4.27) where r is the symmetric rank of X_c (and X_n) and we use the notation $D_{ij} = E_{ij} - E_{ji}$, $i, j = 1, \ldots, N$. E_{ij} is the matrix with entry 1 on line i and column j and 0 otherwise. The results in the complex Grassmannian manifold are obtained further using the exponential map given by Eq. (3.2).

The same result on the calculation of the tangent conjugate locus can be obtained [17] using [12, Proposition 3.1, p.294]. This proposition asserts that $H \in \mathfrak{a}$ is conjugate with o iff $\alpha(H) \in \mathrm{i}\pi\mathbb{Z}^{\star}$ for some root α which do not vanish identically on $\mathfrak{a}$. The eigenvalues of the equation $[H, X] = \lambda X$, $\forall H \in \mathfrak{a}$, $X \in \mathfrak{g}^{\mathbb{C}}$, lead [17] to the values given in Eq. (4.26) for the parameters $t_1 - t_3$.

The direct proof [17] in the Grassmannian manifold uses in Theorem 1 the dependence $Z = Z(B)$ furnished by Eq. (3.2) which gives the geodesics on $G_n(\mathbb{C}^{n+m})$ and Jordan's stationary angles between two n-planes. The stationary angles between two n-planes are given by Lemma 1 and appear in the relation given by Lemma 2.

The proof [17] is done in four steps. (a) First, a diagonalization of the $n \times m$ matrix Z is performed. (b) Second, the Jacobian of a transformation of complex dimension one is computed. (c) The cut locus is reobtained and his contribution to the conjugate locus is taken into account. (d) The nonzero angles are counted using the following property of the stationary angles: if the n' (n)-plane (resp. Z_n) are such that $Z'_{n'} \cap Z_n = Z''_{n''}$, then $n' - n''$ angles of $Z'_{n'}$ and Z_n are different from 0 and n'' are 0. □

5. Conclusion and discussion

In this paper it was shown that for a certain class of homogeneous manifolds which include the symmetric ones there is a relationship between geodesics and coherent states. The starting point [10] of the present investigation, contained in Remark 1, is the observation that for symmetric spaces, if one expresses the parameters Z in Eq. (2.3) as a function of the parameters B in Eq. (2.8), both characterizing the coherent states, explicit local formulas for the geodesic exponential map are obtained. For Hermitian symmetric spaces the dependence $Z = Z(B)$ can be found using the Harish-Chandra decomposition or the so-called Baker–Campbell–Hausdorff formulas [10]. Thus Theorem 1 permits a calculation of the conjugate locus in $G_n(\mathbb{C}^{m+n})$. However, the explicit form of the conjugate locus in $G_n(\mathbb{C}^{m+n})$ is not completely known [17]. The part of the conjugate locus $\boldsymbol{C}_0^W$ determined by Wong is expressible as Schubert varieties [14], while the rest [15] $\boldsymbol{C}_0^I$ can be characterized [17] as the subset of points of $G_n(\mathbb{C}^{m+n})$ which have at least two of the stationary angles with the fixed n-plane $\boldsymbol{O}$ equal. $\boldsymbol{C}_0^I$ contains as subset the maximal set of mutually isoclinic [53] subspaces of the Grassmannian manifold, which are isoclinic spheres [53,54], with dimension given by the solution of the Hurwitz [55] problem. This part referring to the explicit calculation of the conjugate locus on $G_n(\mathbb{C}^{m+n})$ was only briefly included in Section 4, the full details being presented elsewhere [17].

The main remark of this paper contained in Theorem 2, equality (1.1), is a simple consequence of the fact that any manifold is the disjoint union of a maximal normal neighbourhood $\mathcal{V}_0$ of a point 0 and the cut locus $\boldsymbol{CL}_0$. It would be interesting to find a geometrical description of the polar divisor for manifolds which are not characterized by condition (B). On the other side, the problem to find explicitly the cut locus on nonsymmetric spaces is a difficult one [16]. Also it was proved that for homogeneous manifolds satisfying condition (B) and admitting an embedding in an adequate projective Hilbert space a necessary and sufficient condition that a point to belong to the cut locus of another point is that the Cayley distance between the images of the points through the embedding to be $\frac{1}{2}\pi$. This category of manifolds includes all the coherent states manifolds which admit prequantization [19].

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